Waves and solitary pulses in a weakly inhomogeneous Ginzburg-Landau system

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Dynamics of continuous waves (cw's) and solitary pulses (SP's) are considered in the cubic complex Ginzburg-Landau equation with x-dependent coefficients in front of the linear terms, which is a natural model of the traveling-wave convection in a narrow slightly inhomogeneous channel. For the cw, it is demonstrated that even a weak inhomogeneity can easily render all the waves unstable, which may be one of the factors stipulating the so-called dispersive chaos experimentally observed in the convection. Evolution of a SP in the presence of a smooth inhomogeneity is considered by means of the perturbation theory, and it is demonstrated that, in accordance with experimental observations, the spot that is most apt to trap the pulse is the spot with a maximum slope of the inhomogeneity.

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I. INTRODUCTION

Recently, very precise experiments with traveling waves and solitary pulses in a binary-fluid convection have been reported by P. Kolodner and his co-workers [1–3]. The experiments were conducted in a narrow annular channel filled by a water solution of alcohol. It was demonstrated that the solitary pulses (SP's), discovered originally in a motionless state [4], move at a very small velocity if the channel is very homogeneous. However, they can be easily trapped by weak local inhomogeneities. The results presented in Ref. [2] indicate that, roughly speaking, the pulses are apt to be trapped not by a spot where the local overcriticality ϵ (which is, actually, negative when the SP's exist) has a maximum or a minimum, but rather by a maximum of the local slope $\frac{d\epsilon}{dx}$. In Ref. [3] it has been demonstrated that local inhomogeneities of ϵ may also play an important role in the dynamics of continuous waves (cw's), which exist in the binary-fluid convection in the form of traveling waves. In particular, interesting results were produced by interactions of local defects of the cw's (fronts, sources, or sinks) with the local inhomogeneities.

The aim of the present work is to analyze the cw dynamics, as well as that of the SP's, in a weakly inhomogeneous system within the framework of the standard Ginzburg-Landau (GL) models. One can naturally discern between two different types of localized inhomogeneities: ramps and bumps. The ramp is a smooth transient region between two domains with different values of ϵ. As is known, an important property of the ramp is that it accomplishes the wave number selection both in nonwave [5] and in wave [6] models (the former model corresponding to the steady convection in pure liquids). One can also consider a ramp of the *local frequency* ω_0, which was shown to give rise to dynamical chaos in the nonwave GL models [7]. The present work will be focussed on the bump local inhomogeneity, i.e., the spot where ϵ and/or ω_0 are different from their bulk values. An important characteristic of the bump is the ratio of its size to a length of the wave interacting with it (e.g., a length of the SP). In the present work, interaction of the cw's and SP's with a large-scale inhomogeneity will be considered. The GL equations, being, generally, asymptotic equations to govern long-scale modulations of the wave fields [8], should furnish an adequate description of the inhomogeneous system just in this case.

The paper is organized as follows. In Sec. II, the cw solutions of the cubic GL equation with slowly varying parameters are analyzed. It is demonstrated that, if one plugs in realistic values of the coefficients of the GL equation corresponding to the binary-fluid convection, even a weak inhomogeneity may render all the steady cw solutions unstable, which naturally gives way to a dynamical chaos. This is a main result of Sec. II and it may lend a natural explanation to the so-called dispersive chaos observed experimentally in Ref. [1]. Then, in Sec. III, the dynamics of a SP in the inhomogeneous model are considered. To solve the problem analytically, it is this time assumed that the GL equation is close to the nonlinear Schrödinger (NS) equation. Accordingly, the corresponding SP is close to a NS soliton with certain values of its amplitude and velocity. It is demonstrated that, in qualitative accordance with the experimental observations [2], the pulse is apt to be trapped by a local maximum of the inhomogeneity slope. To say this more accurately, as the mean group velocity of the traveling wave deceases, the trapping of the SP first becomes possible at the spot where the inhomogeneity has the steepest slope. With further decrease of the group velocity, the trapping point is shifted away from this spot. Stability conditions for the trapped SP are obtained in an approximate but explicit form.

In the concluding Sec. IV, the results obtained in this paper are briefly summarized, and, in addition, the simplest version of the GL equation, a "dispersive GL equa-

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tion," which may be an adequate model of "dispersive chaos" [1] is highlighted. It is demonstrated that it admits an *exact* SP solution, so that it may be worthy to directly compare this solution with the pulses observed in the binary-fluid convection.

II. TRAVELING WAVES IN THE INHOMOGENEOUS GINZBURG-LANDAU SYSTEM

In this section, the standard cubic GL model will be considered,

$$u_t + cu_x = \epsilon(x)\, u - i\omega_0(x)\, u + (1+i\beta)u_{xx} \\ -(1+i\alpha)|u|^2u, \tag{1}$$

where α and β are the coefficients of the nonlinear and spatial dispersion, c is the group velocity of the long waves in the laboratory reference frame, and $\epsilon(x)$ and $\omega_0(x)$ are the local overcriticality and frequency varying at a large spatial scale. A stationary solution to Eq. (1) is looked for in the usual form,

$$u(x,t) = a(x)\, \exp\left[i\int k(x)\, dx - i\omega t\right], \tag{2}$$

$a(x)$ and $k(x)$ being the local amplitude and wave number, while the frequency ω is assumed to be x independent. Inserting Eq. (2) into Eq. (1), in the lowest approximation (i.e., when gradients of the slowly varying functions are completely neglected) one readily obtains the familiar relations

$$a^2 = \epsilon(x) - k^2; \tag{3}$$

$$k^2 = (\beta-\alpha)^{-1}[\omega - ck - \omega_0(x) - \alpha\epsilon(x)]. \tag{4}$$

Equations (3) and (4) describe a steady cw with the local amplitude and wave number adjusted to the slowly varying coefficients $\epsilon(x)$ and $\omega_0(x)$. Obviously, a necessary condition for stability of this wave is fulfillment of the known cw stability conditions [9] *locally* at each point. In particular, the necessary cw stability condition (see, e.g., Ref. [10]) is

$$a^2(x) \geq 2\epsilon(x)\,(1+\alpha^2)(3+2\alpha^2+\alpha\beta)^{-1}. \tag{5}$$

According to Ref. [1], a reasonable phenomenological model for the cw's in the binary-fluid convection corresponds to the choice $\beta = 0$, with α sufficiently large and negative. As concerns the group velocity c, it was found in the experiments [1] that c was not small for small-amplitude waves, but in the range of the dispersive chaos it practically dropped to zero. In the rest of this section, group velocity will be set $c = 0$.

This choice of the parameters simplifies the analysis. To further simplify it, in what follows below (in this section only) the frequency inhomogeneity is neglected. Then from Eqs. (3) and (4) it ensues that $a^2 = \omega/\alpha$, and finally one obtains from Eqs. (3) through (5) the following relations:

$$\epsilon(x)\frac{2+2\alpha^2}{3+2\alpha^2} < a^2 \equiv \frac{\omega}{\alpha} < \epsilon(x). \tag{6}$$

The left inequality is equivalent to the stability condition (5), while the right one implies that the squared local wave number, related to the amplitude by Eq. (3), must remain positive. It is relevant to emphasize once again that the stability condition (6) is obtained for slowly varying $\epsilon(x)$, and the wave will not necessarily be destabilized if this condition is violated in some narrow region.

If the local overcriticality $\epsilon(x)$ varies between certain limiting values $\epsilon_{\min}$ and $\epsilon_{\max}$, the inequalities (6) give rise to the following restriction:

$$\frac{\epsilon_{\min}}{\epsilon_{\max}} > \frac{2+2\alpha^2}{3+2\alpha^2}. \tag{7}$$

As it was mentioned above, realistic values of the nonlinear dispersion coefficient α can be borrowed from the experimental data presented in Ref. [1]: $|\alpha| \geq 6$. With regard to this, Eq. (7) implies the following restriction:

$$\frac{\epsilon_{\max}-\epsilon_{\min}}{\epsilon_{\max}} < (3+2\alpha^2)^{-1} \leq 0.01. \tag{8}$$

Thus, the degree of the inhomogeneity, measured by the expression on the left-hand side (lhs) of Eq. (8), must be very small to be compatible with existence of the stable stationary cw's. Otherwise, there is no stable cw, and it is natural to expect onset of a dynamical chaos. The experimental results presented in Ref. [1] demonstrate that the so-called dispersive chaos can indeed set in easily in the binary-fluid convection, whose phenomenological governing equation is known to take the form of Eq. (1) with very small β and large α^2. The results obtained in this section suggest that a weak inhomogeneity of the system may be one of the crucial factors triggering the chaos. Therefore, it should be interesting to check experimentally if the onset of the dispersive chaos is sensitive to a weak inhomogeneity of the channel.

III. TRAPPING A SOLITARY PULSE BY A SMOOTH INHOMOGENEITY

A full analytical investigation of dynamics of a SP in an inhomogeneous system is only possible when the GL Eq. (1) may be regarded as a perturbed NS equation. After straightforward transformations, one can rewrite Eq. (1) in the form

$$iU_t + icU_x + U_{xx} + 2|U|^2U = i\epsilon_0(x)U + i\epsilon_1 U_{xx} \\ +\omega_0(x)U, \tag{9}$$

where the nonlinear dissipation present in Eq. (1) is omitted in accordance with the above-mentioned empiric fact [1] that in the effective GL equation for the binary-fluid convection the nonlinear dispersion is much stronger than the nonlinear dissipation. If necessary, the nonlinear dissipation can be readily incorporated into the analysis below.

The SP solution of Eq. (9) with $c = 0$, ϵ_0=const, and ω_0 = const can be found in an exact form [11]. Strictly speaking, this solution is unstable, as it represents a SP over the trivially unstable background value $U = 0$. However, this instability is usually ignored. Indeed, a fully stable SP can be produced by the quintic GL equation [14], and one may hope that, at least in the regime close to the NS equation, the dynamics of the SP in the slightly inhomogeneous quintic model should not be qualitatively different from what will be obtained below for the cubic model.

In the case of small ϵ_1 [i.e., when Eq. (9) is close to the NS equation], the SP solution of Eq. (9) with constant coefficients and with $c = 0$ can be approximately represented in the form of a slightly modified NS soliton [12]:

$$U_{\text{sol}} = 2i\eta_0 \,\text{sech}(2\eta_0(x-\xi)) \exp[i(4\eta_0^2 t + \phi(x-\xi))] \,, \tag{10}$$

where ξ is the coordinate of the soliton center, the equilibrium amplitude of the soliton is

$$\eta_0 = \tfrac{1}{2}\sqrt{3\epsilon_0/\epsilon_1} \,, \tag{11}$$

and the perturbation-induced correction to the soliton phase, which plays a crucial role in various problems [12], is

$$\phi(z) = \tfrac{2}{3}\epsilon_1 \ln\left(\text{sech}(2\eta_0 z)\right) . \tag{12}$$

The next step is to derive perturbation-induced evolution equations for the amplitude η and velocity $V \equiv \frac{d\xi}{dt}$ of a slowly moving and evolving soliton. The easiest way is to use the so-called balance equations [13] for the "mass"

$$M \equiv \int_{-\infty}^{+\infty} |U|^2 dx = 4\eta \tag{13}$$

and momentum

$$P \equiv i\int_{-\infty}^{+\infty} U U_x^* dx = 2\eta V \tag{14}$$

of the soliton. In the course of this derivation, it is necessary to take into account the underlying assumption, according to which the coefficients ϵ_0 and ω_0 vary at a spatial scale which is much greater than the soliton width η^{-1}. Finally, simple calculations lead to the following system of equations:

$$\frac{d\eta}{dt} = 2\epsilon_0\eta - \tfrac{8}{3}\epsilon_1\eta^3 - \tfrac{1}{2}\epsilon_1\eta V^2 + \tfrac{1}{2}\epsilon_1 c\eta V \,, \tag{15}$$

$$\frac{dV}{dt} \equiv \frac{d^2\xi}{dt^2} = -\tfrac{16}{3}\epsilon_1\eta^2(V-c) - 2\omega_0' - \tfrac{4}{3}\epsilon_1\epsilon_0' \,, \tag{16}$$

where the prime stands for $\frac{d}{dx}$, and the values of the functions ϵ_0 and ω_0 and their derivatives are taken at $x = \xi$. Actually, Eq. (15) is the mass-balance equation in its standard form [13], while Eq. (16) is an effective Newton's equation of motion including three forces applied to the soliton which is regarded as a quasiparticle: a driving force restoring the equilibrium value $V = c$ of the velocity in the homogeneous medium; a potential force proportional to ω', and an additional force produced by an interaction of the phase (12) with the gradient of the overcriticality parameter ϵ_0.

A pinned SP corresponds to a fixed point of Eqs. (14) and (15): $V = 0$, $\frac{d\eta}{dt} = 0$. First of all, the fixed point of Eq. (15) recovers the expression (11) for the equilibrium amplitude. Next, with regard to this, Eq. (16) yields

$$\left(\omega_0 + \frac{2}{3}\epsilon_1\epsilon_0\right)' = 2\epsilon_0 c\,. \tag{17}$$

Equation (17) determines at which point the SP (soliton) will be pinned. Note that the smallness of the gradients of the slowly varying coefficients on the lhs of Eq. (16) should be equilibrated by the smallness of the group velocity c. Evidently, the maximum pinning force is exerted at a point where the slope of the inhomogeneity has a maximum. Thus, with the decrease of the group velocity c (experimentally, this may be realized by means of gradually changing the Rayleigh number), the SP is going to be trapped for the first time at the maximum-slope point, which seems to agree with experimental observations [1]. Evidently, at smaller c, Eq. (17) determines, generally speaking, two pinning points with smaller values of the slope, among which one is unstable and one may be stable (see below). Finally, at $c = 0$ the pinning points coincide with the points of zero slope.

Next, it is necessary to analyze stability of the fixed point. One should make use of the smallness of the gradients of ϵ_0 and ω_0 in the stability analysis. In the zeroth approximation, in which the gradients are completely neglected, it is easy to see that linearization of Eqs. (15) and (16) in a vicinity of the fixed point gives rise to three values of the instability growth rate γ: $\gamma_{1,2} = -4\epsilon_0 \pm 2c\sqrt{\epsilon_0\epsilon_1}$ and $\gamma_3 = 0$. The stability condition following from the expression for $\gamma_{1,2}$ is

$$4\epsilon_0 > c^2\epsilon_1\,. \tag{18}$$

In the next approximation, the root γ_3 is no longer zero, and the requirement that it must be negative gives rise to an additional stability condition involving gradients of the slowly varying coefficients. After some algebra, this condition can be cast into the following form:

$$\left(\omega_0 + \frac{2}{3}\epsilon_1\epsilon_0\right)'' > 2c\epsilon_0'\,. \tag{19}$$

Note that, due to the smallness of c, the inequality (18) can be regarded as trivially fulfilled, and then Eq. (19) remains the only actual stability condition for the pinned SP. It is natural to expect that, in the general case, this condition holds for one pinning point, and does not hold for the other one.

IV. CONCLUSION

In this work, dynamics of the continuous waves and solitary pulses were analyzed in the framework of the cubic Ginzburg-Landau model with a smooth inhomogeneity. It was shown that even a very mild inhomogeneity is apt to destabilize all the stationary waves, thus giving way to a dynamical chaos. Trapping of a solitary pulse by the inhomogeneity was also analyzed in detail.

In conclusion, it seems relevant to discuss once again some properties of the general model (1) with the parameters fitted to the empiric data obtained for the binary-fluid convection in the narrow channel [1]. As it was mentioned above, in the lowest approximation this reduces to neglecting the spatial dispersion, nonlinear dissipation, and, at least in certain cases, the group velocity. Thus one arrives at the parameter-free equation, which may be called the "dispersive Ginzburg-Landau equation,"

$$u_t = u + u_{xx} - i|u|^2 u\,. \tag{20}$$

Considering the limit of Eqs. (3), (4), and (5) at $\alpha \to \infty$, $\beta = 0$, it is easy to find that all the cw solutions of Eq. (20) are unstable (these solutions may have only the wave numbers $k = \pm 1$, but their amplitude is arbitrary). Therefore, dynamics governed by this equation may be purely chaotic (from this viewpoint, the results obtained in Sec. II imply that a weak spatial inhomogeneity can easily destruct a narrow cw stability zone remaining at large but finite values of α^2).

Recall that an exact solitary-pulse solution for the general Eq. (1) (with $c = 0$) was found in Ref. [11]. The corresponding solution for Eq. (20) can be obtained from that general solution by means of a limiting procedure,

$$u = (3\sqrt{2})^{1/2}(\mathrm{sech}x)^{1-i\sqrt{2}}e^{-i2\sqrt{2}t}\,. \tag{21}$$

This solution is unstable within the framework of Eq. (20). Nevertheless, one may expect that the actual shape of the experimentally observed SP can be close to that predicted by Eq. (21), which calls for a direct experimental verification.

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[1] P. Kolodner, J.A. Glazier, and H.L. Williams, Phys. Rev. Lett. **65**, 1579 (1990); J.A. Glazier, P. Kolodner, and H. Williams, J. Stat. Phys. **64**, 945 (1991).
[2] P. Kolodner, Phys. Rev. A **44**, 6448 (1991); **44**, 6466 (1991).
[3] P. Kolodner, Phys. Rev. A **46**, 6431 (1992); **46**, 6452 (1992).
[4] J.J. Niemela, G. Ahlers, and D.S. Cannell, Phys. Rev. Lett. **64**, 1365 (1990); K.E. Anderson and R.P. Behringer, Phys. Lett. A **145**, 323 (1990).
[5] L. Kramer, E. Ben-Jacob, H. Brandt, and M.C. Cross, Phys. Rev. Lett. **49**, 1891 (1982); Y. Pomeau and S. Zaleski, J. Phys. (Paris) Lett. **44**, L135 (1983).
[6] B.A. Malomed, Phys. Rev. E **47**, R2257 (1993).
[7] B.A. Malomed and A.B. Rovinsky, Phys. Rev. A **40**, 1640 (1989); B.A. Malomed and A. Weber, Phys. Rev. B **44**, 875 (1991).
[8] M.C. Cross and P.C. Honenberg, Rev. Mod. Phys. **65**, 851 (1993).
[9] T. Stuart and R. DiPrima, Proc. R. Soc. London, Ser. A **362**, 27 (1978).
[10] B.A. Malomed, Z. Phys. B **55**, 241 (1984).
[11] N.R. Pereira and L. Stenflo, Phys. Fluids **20**, 1733 (1977).
[12] B.A. Malomed, Phys. Rev. A **44**, 6954 (1991).
[13] Yu.S. Kivshar and B.A. Malomed, Rev. Mod. Phys. **61**, 763 (1989).
[14] B.A. Malomed, Physica D **29**, 155 (1987); O. Thual and S. Fauve, Phys. Rev. Lett. **64**, 282 (1990).